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On the Degeneracies of Quantum Systems

J. S. DOWKER

Department of Theoretical Physics, University of Manchester, Manchester, England

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We consider the quantum mechanics of a system whose configuration space, $\mathscr{M}$, possesses a transitive group of motions, G. $\mathscr{M}$ can then be identified with the homogeneous space G/H where H is a subgroup of G. For simplicity we assume that G is compact and semi-simple. $\mathscr{M}$ can then be endowed with a G-invariant Riemannian metric and it is reasonable to assume that Schrödinger's equation takes the covariant form

$$-\tfrac{1}{2}\Delta_2\,\psi(q) = i\dot{\psi}(q), \qquad q \in \mathscr{M} \tag{1}$$

where Δ_2 is the Laplace–Beltrami operator, so that the quantum system is invariant under G. Of course this last condition does not fix Schrödinger's equation uniquely.

According to the Peter–Weyl theorem any function on G, say $\tilde{\psi}(g)$, can be expanded in the representation matrices, $\mathscr{D}^{(l)}_{mn}(g)$, thus

$$\tilde{\psi}(g) = \sum \tilde{\psi}^{mn}_{(l)}\,\mathscr{D}^{(l)}_{mn}(g)$$

If we now integrate out the subgroup H we have the expansion of a function on $\mathscr{M}$, which can be considered to be a function on G constant on right cosets, i.e.,

$$\tilde{\psi}(gh) = \tilde{\psi}(g) \equiv \psi(q)$$

Thus

$$\psi(q) = \sum \psi^{mn}_{(l)}\,Y^{(l)}_{mn}(q)$$

where

$$Y^{(l)}_{mn}(q) \equiv Y^{(l)}_{mn}(g) \equiv \int_H \mathscr{D}^{(l)}_{mn}(gh)\,dh = Y^{(l)}_{mn}(gh) \tag{2}$$

The $Y^{(l)}_{mn}$ are the spherical functions on $\mathscr{M}$ introduced by Cartan (1929) in a classic paper (see Vilenkin, 1968). It is important to know how many independent such functions there are for a given set of representation labels (l). This number can be found, in general terms, as follows. From (2) we have

$$Y^{(l)}_{mn}(g) = \mathscr{D}^{(l)}_{mk}(g)\int_H \mathscr{D}^{(l)}_{kn}(h)\,dh$$

and for this to be non-zero the representation (l) of G must contain the trivial, i.e. unit, representation of H at least once, when G is reduced to H. Call the number of times this representation is contained in (l), $n(l)$, then the number of linearly independent $Y_{mn}^{(l)}$ for a given (l) is $n(l)d(l)$ where $d(l)$ is the dimension of the (l)-representation. This is easily shown by reducing the representation (l), of G, to H. The number $n(l)$ is of course the number of independent vectors in the carrier space of the (l)-representation invariant under H (Vilenkin, 1968). Standard group theory provides a general expression for $n(l)$,

$$n(l) = |H|^{-1} \int_H \chi^{(l)}(h)\, dh$$

where $\chi^{(l)}$ is the character of the (l)-representation, and $|H|$ the volume of H.

Turning now to the Schrödinger equation (1) and its eigenfunctions we note that the Laplace–Beltrami operator on $\mathscr{M}$ is given by the restriction of the Laplace–Beltrami operator on G, $\Delta_2{}^G$, to its action on functions on G constant on right cosets. Now $\Delta_2{}^G$ is just the (first) Casimir operator of G and we can now easily determine the energy eigenvalues if the Schrödinger equation is as in equation (1). We find for these eigenvalues

$$E_{(l)} = \tfrac{1}{2}(\mathbf{K}^2 - \mathbf{R}^2)$$

where $\mathbf{K}$ is a vector determined entirely by the representation labels (l) and $\mathbf{R}$ is half the sum of the positive roots of the Lie algebra of G (see, e.g., Racah, 1951). The corresponding eigenfunctions are $Y_{mn}^{(l)}$, up to a normalisation, and so the degeneracy of the $E_{(l)}$ level is $n(l)d(l)$.

References

Cartan, E. (1929). *Rendiconti del Circolo matematico di Palermo*, **53**, 217.

Racah, G. (1951). 'Group Theory and Spectroscopy', lectures given at Princeton University.

Vilenkin, N. J. (1968). *Special Functions and the Theory of Group Representations.* American Mathematical Society, Providence.